Locally $GQ(t,t)$-graphs

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Generalized quadrangle $GQ(s,t)$ is a geometry of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines (with $s > 0$, $t > 0$) and for any antiflag (P,y) there is the unique line z containing P and intersecting y. A hyperoval of $GQ(s,t)$ is regular subgraph of degree $t+1$ without triangles.

Consider locally $GQ(s,t)$-graphs. Locally $GQ(2,t)$-graphs are classified in [1]. The classification of locally $GQ(3,t)$-graphs is finished in [2].

The case $s > 3$ is very difficult. Amply regular locally $GQ(4,2)$-graphs are classified in [3]. In this talk we discuss the problem of description of amply regular locally $GQ(t,t)$-graphs for $t \in \{4,5\}$. Every $GQ(4,4)$ is isomorphic to the classical quadrangle $W(4)$. Every known $GQ(5,5)$ is isomorphic to the $W(5)$ or other classical quadrangle $Q_4(5)$.

Let Γ be a locally $GQ(s,t)$-graphs. Then for every two vertices u,w at distance 2 a subgraph $\Gamma(u) \cap \Gamma(w)$ is a hyperoval. A classification of hyperovals in known classical quadrangle $W(4)$, $W(5)$ and $Q_4(5)$ was obtained by computer calculations in GAP.

Theorem 1. *Amply regular locally $GQ(4,4)$-graph does not exist.*

Theorem 2. *Let Γ be a connected amply regular locally $GQ(5,5)$-graph. Then $\mu \in \{20,26,30,52\}$ and one of the following holds:*
(1) $d(\Gamma) = 2$, *Γ has parameters $(532,156,30,52)$ and eigenvalues 4, -26 of multiplicities 455, 76;*
(2) $d(\Gamma) = 4$ *and $\mu = 20$;*
(3) $d(\Gamma) = 3$.

Corollary. *Let Γ be a connected amply regular locally $GQ(5,5)$-graph, in which for every vertex a the subgraph $\Gamma(a)$ is $W(5)$ or $Q_4(5)$. Then $d(\Gamma) = 2$ and Γ is locally $W(5)$-graph with parameters $(532,156,30,52)$.*

For the vertex set S of the graph Γ we set

$$\Gamma(S) = \cap_{a \in S}(\Gamma(a) - S). \tag{1}$$

A graph Γ is called t-izoregular, if for every $i \leq t$ and for every i-vertex subset S the number $|\Gamma(S)|$ is depend only from isomorphic type

of subgraph induced by S. A graph on v vertices is called absolute izoregular, if it is $(v-1)$-izoregular. t-izoregular graph Γ is called exactly t-izoregular, if it is not $(t+1)$-izoregular. Cameron [4] proved that every 5-izoregular graph Γ is absolute izoregular and is isomorphic pentagon, 3×3-grid, complete multipartite graph $K_{n\times n}$ or its complement. Further every exactly 4-izoregular graph is pseudogeometric for $pG_r(2r, 2r^3 + 3r^2 - 1)$ or its complement. Let $Izo(r)$ be a pseudogeometric graph for $pG_r(2r, 2r^3 + 3r^2 - 1)$. For $r = 1$ we have the point graph of $GQ(2, 4)$, and for $r = 2$ we have MacLaughlin graph.

Hypothesis. *A graph $Izo(r)$ for $r > 2$ does not exist.*

For every vertex a of a graph $Izo(r)$ the subgraph $\Gamma(a)$ is pseudogeometric for $pG_{r-1}(2r-1, r^3 + r^2 - r - 1)$. Makhnev [5] proved that pseudogeometric graph for $pG_{r-1}(2r-1, r^3 + r^2 - r - 1)$ does not exist for $r = 3$, so Hypothesis is valid for $r = 3$. In this talk we consider $Izo(4)$.

Theorem 3. *The following hold:*
(1) *if Γ is strongly regular graph with parameters* $(3159, 1408, 532, 704)$, *a is a vertex of Γ and $\Sigma = \Gamma(a)$, then every 6-clique of Σ contains in some 8-clique of Σ;*
(2) *if Σ is strongly regular graph with parameters* $(1408, 532, 156, 228)$, *in which every 6-every 6-clique contains in some 8-clique, then for every two adjacent vertices b, c of Σ the subgraph $[b] \cap [c]$ is a point graph of $GQ(5, 5)$.*

Proposition. *Let Λ be a strongly regular locally $GQ(5, 5)$-graph with parameters* $(532, 156, 30, 52)$. *Then Λ has not vertex a with $\Lambda(a)$ isomorphic to the point graph of $W(5)$ or $Q_4(5)$.*

References

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